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APPLIED MODEL THEORY AND METAMATHEMATICS.

AN ABRAHAM ROBINSON MEMORIAL PROBLEM LIST

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ABSTRACT

To commemorate ten years since Abraham Robinson's death, open problems in various areas of mathematics are gathered. These problems were chosen because extensions of methods established by Robinson may help contribute to their solution.

Abraham Robinson was born in Wallenburg, Germany in 1918 and died in New Haven, Connecticut in 1974. During his life (spent in Palestine, England, Canada, Israel and the United States) he established ideas and directions whose importance are now well recognized and which continue to serve as guides and sources of inspiration. Certainly every model theorist owes him a large intellectual debt.

A complete scientific biography of Robinson is in preparation by J. Dauben and should appear shortly. For shorter biographies see Macintyre [21], Seligman [42], and Mostow [24].

Among Robinson's achievements: He was one of the founders of model theory and some of its most basic theorems are due to him. He was a first-rate applied mathematician. He made important contributions to the philosophy of mathematics. He is perhaps best well known as the creator of non-standard analysis. (He still remains unsurpassed as a practitioner of the method.)

Perhaps most important of all, he established a way of thinking; of looking at model theory not only as a framework for mathematics, but as a theory which can contribute penetrating specific results in a wide variety of subjects. He himself made contributions across a dazzling spectrum of mathematical specialties.

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As a result, it was his work that, historically, was the most influential in convincing mathematicians that logic belonged firmly in the mainstream of their discipline rather than a subject which, while it might have important things to say about their subject, really was a peculiar and vaguely suspicious subarea of philosophy.

In this paper are listed a set of open problems which the author believes Robinson would have found interesting. The criteria for inclusion of these problems were twofold: (1) they should be intrinsically interesting (as opposed to clearly technical — this is an esthetic distinction rather than a formal one); (2) there should be some reason to believe that the lines of research established by Robinson should help lead to a solution. Of course the choice reflects the author's interests and prejudices. It is hoped that other appropriate problems will be suggested and further lists compiled.

A common response during the gathering of these questions was "The real question is — what methods would Robinson have come up with by now?". However, any attempt to predict Robinson's intuitions is futile. Mathematicians (as well as people in general) may be divided into those whose reasoning is anticipable, those whose ideas seem natural after the fact, and those whose ideas continue to surprise and delight. Robinson was clearly of the third sort even in ordinary conversation.

Moreover, Robinson lived up to A. Fraenkel's designation as a major counterexample to the theory that mathematicians cease having original ideas after their chronological youth. Non-standard analysis was apparently first conceived around 1960 when Robinson was over forty, and his work remainded penetrating and inventive up to his death.

Robinson, during his last illness, wrote to Godel [24] that his philosophy vis-a-vis man's personal immortality was "deliberately incomplete". Immortality of ideas, though, is a concept that he would not have quarreled with. His best memorial then is the ongoing one of research in extensions of areas he helped establish. It is hoped that this problem list will contribute to that memorial.

PROBLEM LIST

1. Bounds in algebra

Let $p_1(x_1, x_2, \ldots, x_n), \ldots, p_k(x_1, \ldots, x_n)$ *be polynomials over a field K and suppose that the radical of the ideal they generate can itself be generated by* $k-1$ *polynomials. Is there a bound on the degrees of these* $k-1$ *polynomials in terms of the degrees of* $p_1, \ldots, p_k$?

Robinson of course was extremely interested and involved in using model theory to produce bounds for algebraic problems. This thread in his work began with his earliest papers and can be seen clearly in his last (posthumous) work *Algorithms in algebra* [31]. The above problem due to van den Dries is just one of many suggested by the paper of van den Dries and Schmidt [7]. This is a paper that Robinson would have admired.

Another problem on bounds (suggested by D. Saltman) comes from a different domain:

Let D be a division algebra finitely generated over the prime field. Is there a uniform bound on the dimension over its center given a bound on the exponent?

The exponent is the minimal number of times the algebra must be tensored with itself to obtain the matrix ring. See Saltman [40] for general background.

2. Diophantine geometry

State and prove a theorem generalizing both the Mahler–Skolem theorem and Siegel's theorem.

This problem is due to H. Furstenberg.

The idea here is to apply techniques of non-standard diophantine geometry. This may be appropriate since both theorems mentioned above should have non-standard proofs.

Siegel's theorem [35] states that curves of genus $\geqq 1$ have only a finite number of integral solutions while curves of genus 0 have an infinite number of integral solutions only in certain describable specific situations. The positive genus case was given a non-standard analysis proof by Robinson and Roquette [35].

The Skolem–Mahler theorem [22] [45] states that if a sequence of natural numbers is defined by a simple recursion of the sort $a_n = \sum_{k=1}^{M} c_k a_{n-k}$ where $c_1, \ldots, c_M$ are integers, then $\{n \mid a_n = 0\}$ is, modulo a finite set, given by a finite union of arithmetic progressions.

This theorem is proved by "the method of Skolem" [45] [46] which involves redefining the sequence a over the p-adics for p sufficiently large and then applying analytic methods to the p-adics. *It would be both reasonable and interesting to have a non-standard proof of this result, as well.* (Although it has not been checked historically, it seems plausible to suppose that this circle of ideas is what led Skolem to define non-standard models of arithmetic.)

These two theorems may be considered related. Suppose $a_n = \sum_{k=1}^{M} c_k a_{n-k}$ as above. Then the formal power series $\sum a_n x^n$ is a rational function of x. Thus the Skolem–Mahler theorem says something about the coefficients of the formal power series of certain kinds of rational functions in one variable.

Now suppose that $p(x, y)$ is a polynomial. Consider the formal power series in two variables $\sum_{m,n} p(m, n) x^m y^n$. Once again, this is a rational function in x and y. (To prove this, note that it suffices to prove this for $p(m, n)$ a monomial.)

However, in general, the coefficients of a rational function need not be given by a polynomial even for one variable. (For example

$$\frac{1}{1 - x - y} = \sum_{i,j} \binom{i+j}{i} x^i y^j$$

and $\binom{i+j}{i}$ is not a polynomial function of i and j.)

So Siegel's theorem also says something about the coefficients of the formal power series of certain rational functions.

Thus a result that would include both Siegel's theorem and the Skolem–Mahler result would state that if $\sum a_{n,m} x^m y^m = f(x, y)$ is a rational function of x and y and if $a_{m,n} = 0$ for infinitely many pairs (m, n) then this set of zeros has a special ("predictable") form and $f(x, y)$ must satisfy some condition which puts in evidence the fact that infinitely many coefficients vanish.

Note. There are many other problems in this area which might benefit from a non-standard approach. A very interesting development would be a non-standard proof of Faltings' theorem [8] [3] (every curve of genus $\geqq 2$ has only a finite number of rational roots). Note that a model-theoretic proof could establish the existence of bounds — a fact that is not known from Faltings' original proof. However the author does not know of a specific line to follow.

3. Definable sets of reals

Is there a non-trivial expansion of the structure $\langle R, +, \cdot, < \rangle$ of the reals for which the definable sets are "simple"?

A set could be called "simple" if, for example, it has only finitely many components in the order of the model. Of course every definable set in $\langle R, +, \cdot, < \rangle$ has this property. In the terminology of Pillay and Steinhorn [29] this means that the search is for an O-minimal expansion of the reals. Other weaker interpretations of "simple" may also be possible.

Van den Dries' candidate is to add exponentiation to the structure. See [6].

4. Banach spaces

Let B be any non-separable Banach space.

(1) *Is there a non-separable closed linear subspace X such that B/X is non-separable?*

(2) *Is there a closed linear subspace X such that B/X is infinite dimensional separable?*

These problems are considered important ones in the geometry of Banach spaces. (See Rosenthal's paper [36] for general background.) In order to understand the connection with model theory first note that in this context "separable" corresponds to "countable".

Hence, for example, a counter-example to problem (1) would require constructing a "large" Banach space with few "large" subspaces — i.e. any "large" subspace should be almost the entire space. Problems in this vein are familiar to model theorists and various tools might be tried. For example, it might be possible to build the space as an $\aleph_1$-union of separable spaces, at each stage taking care not to let the "wrong" spaces grow by using some variant of finite Robinson forcing or some set theoretic combinatorial tool (e.g. $\diamond$). Shelah has a result in this style on a related Banach space problem [43].

The general opinion in Banach space circles is that (2) is probably true while (1) might be false.

ADDED NOTE. Shelah has a counter-example to (1). His construction uses $\diamond$. See *An uncountable construction*, Isr. J. Math. (to appear).

5. Measure theory

Suppose μ is a measure on σ-algebras S_1 and S_2. When can μ be extended to a measure on the σ-algebra generated by S_1 union S_2?

This is a classic problem in measure theory. It is closely related to the bi-measure problem [18]: *Suppose $\mathcal{A}$ and $\mathcal{B}$ are σ-algebras and μ is a function such that*

(i) *for all A in $\mathcal{A}$, $\mu(A, -)$ is a measure on $\mathcal{B}$,*

(ii) *for all B in $\mathcal{B}$, $\mu(-, B)$ is a measure on $\mathcal{A}$.*

When is μ a measure on the product space?

For some sufficient conditions see Kluvanek [18] and Horowitz [13].

Loeb [20] used non-standard analysis to give methods of transforming a *-measure to a standard measure. His general ideas may be applicable to these problems (especially the first one).

Both of these problems are closely related to work on two-parameter Martingales and two-parameter stochastic processes which are related, for example, to Brownian motion on a sheet. *It might be fruitful to* follow up on the work of Anderson [2] and *develop non-standardly these theories as well.* For further results in the one-parameter case see the well-written memoir by Keisler [16] and its bibliography and the articles by Kosciuk, Lindstrom and Perkins in [14].

6. Differentially closed fields

Classify the countable differentially closed fields (or show that there is no classification).

Robinson invented differentially closed fields and apart from his original motivation it has served as a natural source of problems for stability theory. Work of Shelah [44] indicates that there is no classification for the uncountable case.

7. Group theory

Suppose G, H are groups such that for some finite n, G^n (the nth Cartesian product) is elementarily equivalent to H^n. Does it follow that G and H are elementarily equivalent?

This problem, suggested by G. Sabbagh, may be asked for any sort of structure. Mati Rubin has pointed out that it is true for linear order, Boolean algebras, unary predicates and abelian groups. However, it does not hold in general as the following example of Rubin shows:

Let $G = \langle \{0,1,2,\dots\}, f(x) = x+1\rangle$, *let H be two disjoint copies of G. Then* $G^2 \equiv H^2$ *but* $G \not\equiv H$.

It seems most likely that this does not hold for groups — a counter-example might be developed by encoding Rubin's example.

8. Free groups

Are all free groups on a finite number ($\geqq 2$) *of generators elementarily equivalent?*

This is Tarski's classic problem [47]. The language is $(\cdot, {}^{-1}, e)$. The following approach to a solution using non-standard models is due to Charles Mills.

First establish two facts. (In the following F_n is the free group on n generators; a, b, c are generators.)

(i) For any formula φ there is a natural number, n, such that for primes p greater than n, $F_2 \models \varphi(a,b) \Leftrightarrow F_2 \models \varphi(a,b^p)$.

(ii) For any formula φ there is a natural number n and a word w in a and b such that for primes p greater than n, $F_3 \models \varphi(a,b,c) \Leftrightarrow F_3 \models \varphi(a,b^p,w(a,b))$.

In order to prove (i) construct a submodel of $^*F_2$ as follows: Let $C_0 = \{a,b\}$, $C_1 = \{a^m b^n \mid m, n \in {}^*Z\}$, $C_{n+1} = \{x^m y^m \mid m, n \in {}^*Z, x, y \in C_n\}$. Let $C = \bigcup C_n$. Now assume $^*Z = {}^*\langle Z, +\rangle$ was chosen so that there is a prime in Z such that the map $1 \leftrightarrow p$ is an automorphism of Z. (This can be done if , e.g., $^*Z$ is saturated.) This can then be extended to an automorphism of C, by sending $b \leftrightarrow b^p$. If $C < {}^*F_2$ then this would prove (i).

Note that $C < {}^*F_2$ is a plausible conjecture. $C - F_2$ consists of all words with infinite exponents but finite depth — so if a uniform bound on Skolem functions for F_2 could be established this would be true. (This question is in itself interesting and has applications in a wider context.)

In any case, assuming (i) and (ii) are true, here is how to build models $A \models \mathrm{Th}(F_2)$, $B \models \mathrm{Th}(F_3)$ with $A \cong B$.

Let A_0 be the free group on $\langle a, b_0 \rangle$. Let $\varphi_0(a, b_0)$ be a formula on any finite subset of $\langle a, b_0 \rangle$. By (i) find a p appropriate for φ_0. Now attach an element b_1 so that $b_0 = b_1^p$ (i.e., b_0 should be originally chosen so that it has a p-th root). Let A_1 be $\langle a, b_1 \rangle$. Then $A_1 \models \varphi_0(a, b_1)$ by (i). Now repeat the process for φ_2, etc. Let A be the limit of this process. Then A is a model of $\mathrm{Th}(F_2)$.

The construction of B is analogous to that of A, but at each stage rechoose both b and c so that $b_n = b_{n+1}^p$ and $c_n = w(a, b_{n+1})$. By (ii) B is a model of $\mathrm{Th}(F_3)$ but by construction $A = B$ (since eventually all c are words in some a and b).

9. Profinite groups

Is every subgroup of finite index of a topologically finitely generated profinite group open?

This is a very well known problem in the theory of profinite groups. Partial positive results on this problem may be found in [1], [10] and [27]. The conjecture here is that it is false; the method of approach towards building a counter-example involves the use of non-standard models. (This approach is due independently to Manevitz and van den Dries.)

First note that a profinite group (i.e., an inverse limit of finite groups) is closely related to certain $*$-finite groups which will have the same number of $*$-generators as the profinite group has topological generators. (This was first noted by Robinson [32], developed by Hirshfeld and Manevitz [12], and later the ideas were exploited in another situation by Herfort and Manevitz [11].) There is also a close correspondence between, e.g., open subgroups of the profinite group and internal subgroups of the $*$-finite group. It follows from the above that there is a homomorphism π from the $*$-free group on two generators ${}^*F_2$ onto the free profinite group on two generators $\hat{F}_2$.

Results of Sacerdote [38] and Rips [30] show that F_2 and F_3 are π_3 equivalent. Thus by Keisler's sandwich theorem there is a diagram of maps as follows:

$$\begin{array}{ccccccc} F_2 & \xrightarrow{\ i\ } & F_3 & \xrightarrow{\ j\ } & {}^*F_2 & \xrightarrow{\ \sigma\ } & {}^*F_3 \\ & & & & \downarrow{\scriptstyle \pi} & & \\ & & & & F_2 & & \end{array}$$

Here the maps commute with the natural embeddings of F_2 in $^*F_2$ and F_3 in $^*F_3$. It we let a, b be generators for F_2 and $\{a, b, c\}$ be generators for F_3 then we have (where w is a non-standard length word in a and b)

$$a, b \xrightarrow{i} a, b, c \xrightarrow{j} a, b, w \xrightarrow{\sigma} a, b, c.$$

Now define the (internal) map $\varphi : {}^*F_3 \to Z_2$ by $a, b \to 0$, $c \to 1$. It is immediate that $\varphi \circ \sigma$ is external since it sends a to 0 and b to 0 but is nonetheless onto Z_2. Now consider the kernel of $\varphi \circ \sigma$. This is clearly external.

To establish the needed counterexample two facts must be established:

(1) $\pi(\ker(\varphi \circ \sigma))$ is not open in F_2. (This should be relatively easy.)

(2) $\varphi \circ \sigma$ induces an epimorphism, $\varphi \circ \sigma$, from $\hat{F}_2$ to Z_2.

In order to prove (2) it should suffice to show that an infinitesimal of $^*F_2$ (i.e., an element of $\ker \pi$) is in the kernel of $\varphi \circ \sigma$. This is a reasonable request since, e.g., it would hold if σ is constructed so that infinitesimals of $^*F_2$ go to infinitesimals of $^*F_3$ — or even if σ of an infinite product of squares in $^*F_2$ is an infinite product of squares in $^*F_3$. Moreover, examination of the proof of the Keisler sandwich theorem [4] reveals that the construction of σ is quite flexible.

10. Homomorphic model theory

Definite a model theoretic analogue of projectivity and/or freeness.

Model theory has developed mainly over two different categories. For "west coast" theory (in Keisler's terminology [15]) the morphisms are elementary embeddings. For "east coast" model theory the morphisms are monomorphisms. Of course the situation is much more complicated than a simple bifurcation — for example, many of applied model theory's best successes occur in instances where the two theories coalesce.

In algebra, other kinds of maps, e.g. epimorphisms, are just as important but in such contexts model-theoretic tools seem not to be as appropriate. Thus, for example, algebraic closure (model completeness and its cousins) are understood very well but comparable insights into notions such as freeness and projectivity are lacking. In this context, a systematic development of homomorphic model theory would be desirable. It is possible that an approach will require a different sort of language than is usually employed.

In his early work, Robinson [33] [34] dealt with these problems technically by working in the diagram of some huge model and then dealing with homomorphisms as collections ("ideals") of equality relations. (This is why he like to treat equality as a relation rather than as a logical identity.)

Of course we have *some* tools — Lyndon's theorem being the most important. Cherlin, van den Dries and Macintyre [5] obtained some important results in a special circumstance. Their exception proves the rule, though. Their "co-logic" works because their homomorphisms of interest are in a formal duality with that of certain monomorphisms of fields.

G. Sacerdote [37] [39] made an attempt to carry out finite forcing constructions in a homomorphic setting (for groups). More recently, Manevitz and Rowen [23] have used a dual of infinite forcing to give nice uniform definitions of various "generic division rings" (in the sense of polynomial identity ring theory).

Another recent result is that of Oger [25], who has shown that finitely generated finite-by-abelian groups are elementarily equivalent if and only if they have the same finite homomorphic images. However he also has shown [26] that this is not the case for finitely generated nilpotent groups of class 2.

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